where D is a negative integer $\equiv 0 \pmod 4$, or as a root of the equation

$$w^2 + w + \frac{1-D}{4} = 0 \ldots (1'),$$

where D is a negative integer $\equiv 1 \pmod 4$, then the set of complex numbers

$$m + nw \ldots (2),$$

m and n real integers, possesses the notable property of being closed under addition and multiplication just as the set of real integers is.

The theorem concerning the uniqueness of the decomposition of real integers into prime factors is based on the algorithm of the greatest common divisor. This algorithm depends on the fact that for all real numbers ξ it is possible to choose a real integer m such that

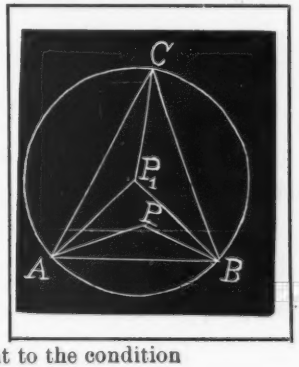
$$| \xi - m | < 1 \ldots (3).$$

The complete formal analogy between the complex number system (2) and the set of real integers leads to the remark: If w be such that for all complex numbers ξ it is possible to determine a complex number $m+nw$ such that

$$| \xi - (m + nw) | < 1 \ldots (4),$$

then the theorem of unique decomposition holds for these number systems. It is the purpose of this note to determine, by a geometrical scheme, for which of these systems condition (4) obtains.

In the complex plane we mark off the network of points $m+nw$, in particular the points $A \equiv 0$, $B \equiv 1$, $C \equiv w$. Since the real part of w is either 0 or $\frac{1}{2}$, the projection of C on AB falls on AB, which is of length 1, and *the triangle ABC is an acute or right triangle.* Further we write $w = r.e^{i\theta}$, whence noting that



$$| w | = \sqrt{-\frac{D}{4}} \text{ or } | w | \sqrt{\frac{1-D}{4}},$$

we obtain $r \geqq 1 \ldots (5)$.

Consider the points Q of the network $m + nw$ and a point P of the plane. At each point P there is clearly a least of the distances PQ; we call this distance $d(P)$. The condition (4) is then equivalent to the condition

$$d(P) < 1 \ldots (6) \text{ for all points } P.$$

In order to give the condition (5) explicit form we construct the circumcenter P_1 of the triangle ABC, so that

$$P_1A=P_1B=P_1C=\rho\ldots(7);$$

also since this triangle is acute or right, P_1 does not lie without the triangle. We first prove that

$$d(P_1)=\rho\ldots(8).$$

By (7) it is clear that if (8) is not true we have $PQ<\rho$ for some Q not A, B, or C; then Q is within the circumscribing circle. Assume if possible such a point Q to exist. From the character of the network it is clear that Q is not within the triangle ABC and accordingly Q must lie on one of the three circular segments, as AB. If then we imagine the network as made up of the intersection of two families of equidistant parallel lines, parallel to CA and CB, it is clear that the line parallel to CA through Q intersects AB produced from B, in β, and that the line parallel to CB through Q intersects AB, produced from A, in α. Thus

$$\angle\beta Q\alpha \geqq \angle BQA\ldots(9).$$

But

$$\angle BQA+\angle ACB>180^\circ\ldots(10),$$

since Q lies in the circle on the opposite side of AB from C. Further it is obvious that $\angle ACB=\angle\beta Q\alpha$. Adding corresponding sides of these equations we infer that, after dividing the inequality by 2, $\angle ACB>90^\circ$, which is not possible as the triangle ABC is an acute or right triangle. Hence (8) is proved.

Consider now an arbitrary point P of the triangle ABC, not P_1; then P lies on one of the triangles AP_1B, BP_1C, CP_1A, say AP_1B, so that $AP+PB<AP_1+P_1B(=2\rho)$. Therefore either AP or $PB<\rho$, whence

$$d(P)<\rho,\ P \text{ not at } P_1\ldots(11).$$

From (8) and (11) we infer that

$$\text{Maximum } d(P)=\rho\ldots(12)$$

for the triangle ABC, and hence for the plane, as the whole plane can be built of such triangles. Thus (6) is equivalent to

$$\rho<1\ldots(13).$$

But $\rho=\dfrac{BC}{2\sin\angle BAC}=\dfrac{\sqrt{(1+r^2-2r\cos\theta)}}{2\sin\theta}$, so that (13) is the same as $1+r^2-2r\cos\theta<2\sin\theta$, or $(r-\cos\theta)^2<3\sin^2\theta$. Noting that from (5), $r\geqq\cos\theta$, we find that

$$r<2\cos(\theta-60^\circ)\ldots(14).$$

This makes $r<2$ so that for systems (1)

$$-\frac{D}{4}=1, 2, 3,$$

and for systems (1'),

$$\frac{1-D}{4}=1, 2, 3,$$

are the possible cases. Investigating these cases using (14), we find that (4) obtains for the equations

$$w^2+1=0;\ w^2+2=0;\ w^2+w+1=0;\ w^2+w+2=0;\ w^2+w+3=0 \ldots (15).$$

In these cases we have a greatest common divisor process.

If we define a prime number N of a system (1) or (1') as one which has no factor n such that $1<|n|<|N|$, we have the theorem:

Every number N of one of the systems (15) can be decomposed essentially in but one way into prime factors. The non-essential variations are obtained frem the factors f such that $|f|=1$, which are the complete units.

The work here given completes the consideration of the case for a negative discriminant D, and agrees with the results of Dedekind, *Zahlentheorie*, fourth edition, page 450.

ON THE CHORD OF CONTACT OF TANGENTS TO A CONIC.

By W. D. LAMBERT, Washington, D. C.

If we wish to find the equation of the chord of contact of tangents drawn from a point $P_1\equiv(x_1, y_1)$, to the conic*

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0 \ldots (1),$$

the straightforward way is to denote the unknown points of contact $P_2\equiv(x_2, y_2)$ and $P_3\equiv(x_3, y_3)$, and proceed to find the values of those coördinates. For this purpose we get two equations by substituting (x_2, y_2) and (x_3, y_3) in (1), and two more by expressing the fact that P_2 lies on the tangents through P_2 and P_3. These are

$$Ax_1x_2+\frac{B}{2}(x_1y_2+x_2y_1)+Cy_1y_2+\frac{D}{2}(x_1+x_2)+\frac{E}{2}(y_1+y_2)+F=0 \ldots (2),$$

*I carry the work through for the general case, but as beginners are likely to be appalled by the mere length of an equation that is simple enough in principle, it is advisable in teaching to take at first a simpler special form of the equation of the conic.

$$Ax_1x_3+\frac{B}{2}(x_1y_3+x_3y_1)+Cy_1y_3+\frac{D}{2}(x_1+x_3)+\frac{E}{2}(y_1+y_3)+F=0\ldots(3).$$

These four equations give x_2, y_2, x_3, y_3 in terms of A, B, C, etc., and x_1, y_1. The required chord of contact is found by substituting their values in the formula for the straight line

$$\frac{y-y_2}{x-x_2}=\frac{y_3-y_2}{x_3-x_2}\ldots(4).$$

This direct solution is obvious in principle, but very tedious in the analytic work, and does not appear in text books. The favorite method in elementary treatises is to *assume* the result, namely, that the equation of the chord of contact of tangents to the conic (1) is

$$Ax_1x+\frac{B}{2}(xy_1+x_1y)+Cy_1y+\frac{D}{2}(x+x_1)+\frac{E}{2}(y+y_1)+F=0\ldots(5),$$

and then prove that (5) is a straight line passing through the points of contact P_2 and P_3. This process is certainly contrary to the spirit of *analytic* geometry, and seems obscure to many beginners.

The direct attack may be made easier by noting that we require not x_2, y_2, x_3, and y_3 themselves, but only certain combinations of them that are easy to find. Subtract (2) from (3) and factor so as to bring out x_3-x_2 and y_3-y_2. We may write the resulting equation in the form

$$\frac{y_3-y_2}{x_3-x_2}=-\frac{Ax_1+\frac{B}{2}y_1+\frac{D}{2}}{\frac{B}{2}x_1+Cy_1+\frac{E}{2}}.$$

Substitute this value in (4) and clear of fractions; after transposing the result may be written

$$Ax_1x+\frac{B}{2}(x_1y+xy_1)+Cy_1y+\frac{D}{2}x+\frac{E}{2}y-[Ax_1x_2+\frac{B}{2}(x_1y_2+x_2y_1)+Cy_1y_2]+$$

$$\frac{D}{2}x_2+\frac{E}{2}y_2=0\ldots(6).$$

Adding the identity (2) to (6), we get the required equation (5).

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

265. Proposed by G. W. GREENWOOD, M. A., Dunbar, Pa.

Obtain the reduced cubic $4\theta^3 - I\theta + J = 0$ of the biquadratic $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$.

I. Solution by the PROPOSER.

Assume $ax^4 + 4bx^3 + 6cx^2 + 4dx + e \equiv a(x^2 + 2mx + p)(x^2 + 2nx + q)$.

$\therefore a(m+n) = 2b$; $a(4mn + p + q) = 6c$; $a(mq + np) = 2d$; $apq = e$.

Let $amn = c - \theta$; then $a(p+q) = 2c + 4\theta$.

Substituting in the identity

$$\begin{vmatrix} 1 & 1 & 0 \\ m & n & 0 \\ p & q & 0 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 0 \\ n & m & 0 \\ q & p & 0 \end{vmatrix} \equiv \begin{vmatrix} 2 & m+n & p+q \\ m+n & 2mn & mq+np \\ p+q & mq+np & 2pq \end{vmatrix} \equiv 0$$

we obtain $\dfrac{8}{a^3} \begin{vmatrix} a & b & c+2\theta \\ b & c-\theta & d \\ c+2\theta & d & e \end{vmatrix} = 0$; *i. e.*, $4\theta^3 - I\theta + J = 0$.

II. Solution by L. E. NEWCOMB, Los Gatos, Cal.

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e \equiv x^4 + \frac{4bx^3}{a} + \frac{6cx^2}{a} + \frac{4dx}{a} + \frac{e}{a} = 0 \ldots (1).$$

Let $x = y/a$; then (1) becomes $y^4 + 4by^3 + 6acy^2 + 4a^2dy + a^3e = 0 \ldots (2)$.

$y^4 + 4by^3 + 6acy^2 + 4a^2dy + a^3e + (ay + \beta)^2 = (y^2 + 2by + \lambda)^2$, by Ferrari's solution; whence, by equating coefficients of like powers of y, and eliminating α and β, $\lambda^3 - 3ac\lambda^2 + a^4(4bd - ae)\lambda - a^3(2d^2 + 2b^2e - 3ace) = 0$. Let $4^{\frac{1}{3}}\theta + ae = \lambda$, then

$4\theta^3 - 4^{\frac{1}{3}}a^2(3c^2 + ac - 4bd)\theta + a^3[c(4bd - ac - 2c^2) - (2cd^2 + 2b^2e - 3ace)] = 0 \ldots (3)$,

which is of the required form.

266. Proposed by L. E. NEWCOMB, Los Gatos, Calif.

Find the nth term and the sum of n terms of the series $1 + 3 + 7 + 17 + \ldots$.

I. Solution by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

Let $u_1, u_2, u_3, u_4, \ldots, u_n$ be the terms of the series, in the Calculus of Finite Differences, and let $\triangle u_1$, $\triangle^2 u_1$, $\triangle^3 u_1$, and $\triangle^4 u_1$ be the symbols for the different orders of differences.

The differences and the terms of this special series may be arranged as follows:

$$\begin{array}{lccccccc} u_1 = & 1 & 3 & 7 & 17 & . & . & . \\ \triangle u_1 = & . & 2 & 4 & 10 & . & . & . \\ \triangle^2 u_1 = & . & . & 2 & 6 & . & . & . \\ \triangle^3 u_1 = & . & . & . & 4 & . & . & . \end{array}$$

This shows that the third difference of u_1 or $\triangle^3 u_1$, is constant; and that, therefore, the fourth, and all higher differences, in this series, must vanish.

We have always, $u_1 = u_1$; and then we have for the next term, $u_2 = u_1 + \triangle u_1 = u_1(1+\triangle)$; and then, $u_3 = u_1 + \triangle u_1 + \triangle(u_1 + \triangle u_1) = u_1 + \triangle u_1 + \triangle u_1 + \triangle^2 u_1 = u_1 + 2\triangle u_1 + \triangle^2 u_1 = u_1(1+\triangle)^2$; and so on for higher orders; and in which $\triangle$ may be considered, first, as a symbol of operation, and second, as a symbol of quantity.

The symbols and operations may now be exhibited as follows, as they conform to the law of the Binomial Theorem:

$$\begin{aligned} u_1 &= u_1 \\ u_2 &= u_1(1+\triangle) \\ u_3 &= u_1(1+\triangle)^2 \\ u_4 &= u_1(1+\triangle)^3 \\ &\ldots\ldots \\ u_n &= u_1(1+\triangle)^{n-1} \end{aligned}$$

Expand the term for u_n and we have for its value:

$$u_n = u_1[1+(n-1)\triangle + \frac{(n-1)(n-2)}{2}\triangle^2 + \frac{(n-1)(n-2)(n-3)}{2\times 3}\triangle^3 + \ldots +]. \ldots (A).$$

Let S_n = the value of the sum of n terms and we have:

$$S_n = u_1 + u_2 + u_3 + u_4 + \ldots + u_n.$$

We also have for the second member:

$$S_n = u_1[1+(1+\triangle)+(1+\triangle)^2+(1+\triangle)^3+(1+\triangle)^4+\ldots+(1+\triangle)^{n-1}] \ldots (B).$$

Sum (B) and we have:

$$S_n = \frac{u_1[(1+\triangle)^n - 1]}{\triangle}.$$

Expand the term, $(1+\triangle)^n$, subtract 1, and divide by $\triangle$, and we have:

$$S_n = u_1[n + \frac{n(n-1)}{2}\triangle + \frac{n(n-1)(n-2)}{3!}\triangle^2 + \frac{n(n-1)(n-2)(n-3)}{4!}\triangle^3 + \ldots] \ldots (C).$$

In (A) and (C) remove the brackets so as to unite the symbols of operation and the symbols of quantity and we have:

$$u_n = u_1 + (n-1)\triangle u_1 + \frac{(n-1)(n-2)}{2!}\triangle^2 u_1 + \frac{(n-1)(n-2)(n-3)}{3!}\triangle^3 u_1 + \ldots (D);$$

and

$$S_n = nu_1 + \frac{n(n-1)}{2}\triangle u_1 + \frac{n(n-1)(n-2)}{3!}\triangle^2 u_1 + \frac{n(n-1)(n-2)(n-3)}{4!}\triangle^3 u_1 + \ldots (E).$$

In (D) and (E) substitute the values, $u_1=1$, $\triangle u_1=2$, $\triangle^2 u_1=2$, and $\triangle^3 u_1=4$, from the problem and its differences, and we have, after reduction:

$$u_n = \tfrac{1}{3}[2n^3 - 9n^2 + 19n - 9] \ldots (F); \text{ and}$$
$$S_n = \tfrac{1}{6}[n^4 - 4n^3 + 11n^2 - 2n] \ldots (G).$$

Equations (F) and (G) are true for all values of n for the special series under consideration. When $n=4$, $u_n = u_4 = 17$, and $S_n = S_4 = 28$, as may be seen by inspecting the series in the problem.

But equations (D) and (E) are perfectly general when the series follows any regular law of progression; as we have to know, only, the value of the leading term, and the leading differences up to the difference that vanishes, to find the value of any term in a series and the sum of that series.

II. Solution by L. E. NEWCOMB, Los Gatos, Cal., and G. W. GREENWOOD, M. A., Dunbar, Pa.

Let $S \equiv u_1 + u_2 x + u_3 x^2 + \ldots$ where $u_1=1$, $u_2=3$, $u_3=7$, and, in general, $u_n = 2u_{n-1} + u_{n-2}$, x, of course, being less than unity, numerically.

$\therefore (1-2x-x^2)S = u_1 + (u_2 - 2u_1)x$; *i. e.*,

$$S = \frac{1+x}{1-2x-x^2} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$$

where $\alpha = 1+\sqrt{2}$, $\beta = 1-\sqrt{2}$, $A = \frac{1}{2}(1+\sqrt{2})$, $B = \frac{1}{2}(1-\sqrt{2})$.

$\therefore u_n = A\alpha^{n-1} + B\beta^{n-1} = \frac{1}{2}[(1+\sqrt{2})^n + (1-\sqrt{2})^n]$.

Let $S_n = u_1 + u_2 + \ldots + u_n$; $S_n(1-2-1) = u_1 + u_2 - 2u_1 - 3u_n - u_{n-1}$; *i. e.*, $S_n = \frac{1}{2}[3u_n + u_{n-1} - 2]$.

Solved in a similar manner by J. Scheffer.

CALCULUS.

219. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Evaluate (a) $\int_0^{\frac{1}{2}\pi} \frac{\sin mx \sin nx}{\sin x}\,dx$; (b) $\int_0^{\frac{1}{2}\pi} \frac{\cos mx \sin nx}{\sin x}\,dx$, where n is a positive integer. Also, modify the result for the case of m an integer.

Solution by S. A. COREY, Hiteman, Iowa.

When n is a positive integer, $\sin nx$ may be developed into a sine power series divisible by $\sin x$. Substituting this development in (a) or (b) each term may be readily integrated. This method is, of course, also applicable when m is an integer in (a), but when (b) m is an integer and n not an integer this method fails. In the latter case, as well as in the other cases, an approximate value of (a) and (b) may be deduced by the use of formula (1), page 12, AMERICAN MATHEMATICAL MONTHLY, January, 1906. By using no term higher than the third (the term involving B_2), and by obtaining $f'(x)$ and $f^{IV}(x)$ by differentiating the right members of the following identities:

$$(c)\ \int \frac{\cos mx \sin nx}{\sin x}dx = \int \frac{\sin(m+n)x}{2\sin x}dx - \int \frac{\sin(m-n)x}{2\sin x}dx,$$

$$(d)\ \int \frac{\sin mx \sin nx}{\sin x}dx = \int \frac{\cos(m-n)x}{2\sin x}dx - \int \frac{\cos(m+n)x}{2\sin x}dx,$$

the following developments are obtained:

$$(a)=\frac{\pi}{2.2.r}\left\{\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}+2\left[\frac{\sin(m\pi/2r)\ \sin(n\pi/2r)}{\sin(\pi/2r)}\right.\right.$$

$$\left.\left.+\frac{\sin(2m\pi/2r)\sin(2n\pi/2r)}{\sin(2\pi/2r)}+\ldots+\sin\frac{(r-1)m\pi}{2r}\sin\frac{(r-1)\,n\pi}{2r}\right]\right\}$$

$$-\frac{\pi^2}{6.2.2!(2r)^2}\left(s\sin\frac{s\pi}{2}-t\sin\frac{t\pi}{2}-2mn\right)$$

$$+\frac{\pi^4}{30.(2r)^4.2.4!}\left[(t^3-3t)\sin\frac{t\pi}{2}-(s^3-3s)\sin\frac{s\pi}{2}+2\,m\,n(m^2+n^2-1)\right]\ldots(1).$$

$$(b)=\frac{\pi}{2.2.r}\left\{\cos\frac{m\pi}{2}\sin\frac{n\pi}{2}+n+2\left[\frac{\cos(m\pi/2r)\ \sin(n\pi/2r)}{\sin(\pi/2r)}\right.\right.$$

$$\left.\left.+\frac{\cos(2m\pi/2r)\sin(2n\pi/2r)}{\sin(2\pi/2r)}+\ldots+\frac{\cos[(r-1)m\pi/2r]\ \sin[(r-1)n\pi/2r]}{\sin[(r-1)\pi/2r]}\right]\right\}$$

$$-\frac{\pi^2}{6.(2r)^2.2.2!}\left[s\cos\frac{s\pi}{2}-t\cos\frac{t\pi}{2}\right]$$

$$+\frac{\pi^4}{30.(2r)^4.2.4!}\left[(3s-s^3)\cos\frac{s\pi}{2}-(3t-t^3)\cos\right]\ldots(2),$$

where $s=(m+n)$, $t=(m-n)$. To insure rapid convergence, let $r>(m+n)$. If in (b), $(m+n)=(2p+1)$, m, n, and p integers, it is readily seen that the value of (b) is zero.

In order to test the accuracy of the work of computation as well as to test the convergence of the series, it is sometimes advisable to find the value of the definite integral with $r=2a$ after its value has been found with $r=a$. The work involved in this test is usually not great as the work that has been done when $r=a$ is made use of when $r=2a$.

To show the rapid convergence of (1) and (2) the two following simple examples will suffice:

$$\int_0^{\frac{1}{2}\pi} \frac{\cos(3x/2)\sin x}{\sin x}dx = \int_0^{\frac{1}{2}\pi} \cos\frac{3x}{2}dx = \tfrac{2}{3}\sqrt{\tfrac{1}{2}}.$$ Here $m=\frac{1}{2}$, $n=1$. Taking

$r=3$, we have $\int_0^{\frac{1}{2}\pi} \frac{\cos(3x/2)\sin x}{\sin x}dx = \frac{\pi}{12}(1+\sqrt{\tfrac{1}{2}}) + \frac{3\pi^2\sqrt{\frac{1}{2}}}{6^3.4} + \frac{7\pi^4\sqrt{\frac{1}{2}}}{30.6^4.2.4!} = .47141.$

Similarly, $\int_0^{\frac{1}{2}\pi} \frac{\sin(3x/2)\sin x}{\sin x} = \frac{\pi}{12}(2+3\sqrt{\tfrac{1}{2}}) + \frac{3\pi^2}{6^3.4}(1+\sqrt{\tfrac{1}{2}}) + \frac{\pi^4}{30.6^4.2.4!}$

$\times(\frac{7}{4})(1+\sqrt{\frac{1}{2}}) = 1.13807$, both results being correct to five decimal places.

Also solved by G. B. M. Zerr.

DIOPHANTINE ANALYSIS.

137. Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.

Prove that all multiply perfect numbers of multiplicity n having only n distinct primes are comprised in $n=2, 3, 4$.

Solution by JACOB WESTLUND, Ph. D., Purdue University, Lafayette, Ind.

If $p_1, p_2, \ldots, p_n$ are the distinct prime factors of a number of multiplicity n, we must have $n < \prod_i^{1,n} \frac{p_i}{p_i-1}$, and hence $n < \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdots\frac{2n-1}{2n-2}$. But this is impossible when $n>4$, as seen by induction. For we have

$$(n+1)1.2.4.6\ldots2n = n.1.2.4.6\ldots2n + 1.2.4.6\ldots2n.$$

Now if $n.1.2.4.6\ldots(2n-2) > 2.3.5.7\ldots(2n-1)$, it follows that

$$(n+1)1.2.4.6\ldots2n > 2.3.5.7\ldots(2n-1)2n + 1.2.4.6\ldots2n, \text{ or}$$
$$(n+1)1.2.4.6\ldots2n > 2.3.5.7\ldots(2n+1) - 2.3.5.7\ldots(2n-1) + 1.2.4.6\ldots2n.$$

Hence $(n+1)1.2.4.6\ldots2n > 2.3.5.7\ldots(2n+1)$. For $n=5$ we have

$$5 > \tfrac{2}{1}\cdot\tfrac{3}{2}\cdot\tfrac{5}{4}\cdot\tfrac{7}{6}\cdot\tfrac{9}{8} = \tfrac{315}{64}.$$

Hence for all values of $n>4$ we have $n > \prod_i^{1,n} \frac{p_i}{p_i-1}$, which proves the theorem.

GEOMETRY.

293 (Incorrectly numbered 290). Proposed by FRANCIS RUST, C. E., Allegheny, Pa.

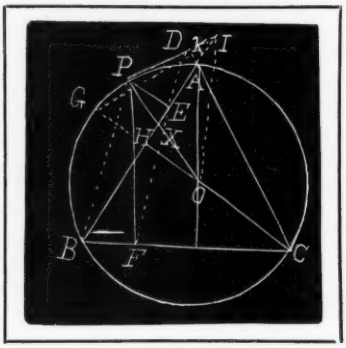
The pedal line of any point on a triangle's circum-circle bisects the distance between this point and the ortho-center of the triangle.

I. Solution by G. W. GREENWOOD, M. A., Dunbar, Pa.

Take, for convenience, an acute triangle ABC and the point P on the arc BC, the feet of the perpendiculars upon BC and CA being D and E, respectively. P is not supposed coincident with B or C. Call the orthocenter O, and *let* BO cut the circle again in G; *let* PQ intersect DE in H and AC in F. Basing the demonstration mainly upon pictorial evidence, we have, since the quadrilateral $PDEC$ is cyclic, $\angle PED = \angle PCD = \angle PGB = \angle GPE$. Hence the triangle PHE is isosceles, and therefore HEF is isosceles, and also $PH = HF$. It can easily be shown that OG is bisected by AC. Hence $\angle OFA = \angle GFA = \angle HFE = \angle HEF$. Hence DE bisects PF and is parallel to OF. It therefore bisects PO.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let P be a point in the circumference of the circle, and PD, PE, PF be the perpendiculars let fall from P upon the three sides of $\triangle ABC$, the *straiyht* line DEF is the pedal for the point P. Let O be the orthocenter, and draw OH perpendicular AB and extend it to G; join G with P and produce it until it meets the pedal DEF at K and the side AB produced at I; draw PB and IO. We now have $\angle PED = \angle PAD = \angle PBC = \angle PGH = \angle IPE$, PE being parallel to GH; hence $\angle KEI = \angle KIE$, being the complements of the equal angles IPE and PED; therefore $PK = KE = IK$. But $\angle HIO = \angle GIH = \angle DEI$. Therefore DX is parallel to IO, X being the point of intersection of PO and the pedal DEF. Since K is the middle point of PI, X must be the middle point of PO. Q. E. D.



294 (Incorrectly numbered 292). Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Apply the locus of $(x^2+y^2)^3 = mx^3$ to the problem of finding a cube m times a given cube.

[No solution has been received.]

295. Proposed by W. J. GREENSTREET, M. A., Editor Mathematical Gazette, Stroud, England.

A variable circle touches an ellipse, and the chord of contact through the other two points of intersection touches a similar coaxial ellipse. Find the locus of the center of the variable circle.

Solution by G. W. GREENWOOD, M. A., Dunbar, Pa.

Let the circle be tangent at the point $P\equiv(a\cos\theta, b\sin\theta)$ to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and let the chord of contact of the other two points touch at Q the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\lambda^2.$$

Since the tangents at P and Q are equally inclined to the axes, the coördinates of Q are $(\lambda a\cos\theta, -\lambda b\sin\theta)$, or $(-\lambda a\cos\theta, \lambda b\sin\theta)$. Since a chord of a conic, tangent to a similar coaxial conic, is bisected at the point of contact, the center of the circle lies on the normal at Q; and also lies on the normal at P; *i. e.*, on

$$\frac{ax}{\cos\theta}+\frac{by}{\sin\theta}=\pm\lambda(a^2-b^2), \text{ and } \frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2.$$

Hence the required locus is

$$\frac{4a^2x^2}{(1\pm\lambda)^2}+\frac{4b^2y^2}{(1\mp\lambda)^2}=a^2-b^2.$$

PROBLEMS FOR SOLUTION.

ALGEBRA.

270. Proposed by GEORGE H. HALLETT, Ph. D., Assistant Professor of Mathematics in The University of Pennsylvania, Philadelphia, Pa.

Find the simplest integral form of the sum $y(y-1)\ldots(y-x)+2y(2y-1)\ldots(2y-x)+\ldots+zy(zy-1)\ldots(zy-x)$.*

271. Proposed by L. E. NEWCOMB, Los Gatos, California.

Sum the series $\frac{a}{b}+\frac{a^3}{3b^3}+\frac{a^5}{5b^5}+\ldots$ to ∞, $b>a$.

272. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Prove that the relations $x=\frac{ar+bs}{\lambda}=\frac{as-br}{\mu}=\frac{a\gamma-b\mu}{r}=\frac{a\mu+b\lambda}{s}$ between the finite real quantities $x, a, b, r, s, \lambda, \mu$ requires that $x^2=a^2+b^2$.

*This series is of frequent occurrence in certain investigations in Group Theory. Ed.

GEOMETRY.

299. Proposed by G. W. GREENWOOD, M. A., Dunbar, Pa.

Show that the circle on any focal radius of an ellipse touches the auxiliary circle.

300. Proposed by J. J. QUINN, Ph. D., Scottdale, Pa.

Trisect an angle by means of a tractrix.

301. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Apply the locus of $r=a(1+2\cos\theta)$ to the trisection of an angle. Describe the curve by continuous motion.

302. Proposed by F. H. SAFFORD, Ph. D., University of Pennsylvania.

Through a given point within a circle draw any two chords, also a radius and a secant perpendicular to the radius. Let the extremities of the chords be taken as vertices of a quadrilateral. Show that the sides of the quadrilateral, produced when necessary, cut the secant in points equidistant, in pairs, from the given point. [A proof by Euclidean geometry is preferred, as the problem was originally given to a high school class.] Must the given point be within the circle?

CALCULUS.

228. Proposed by B. F. FINKEL, Ph. D., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A sphere, radius r, is dropped into a conical vessel whose vertex angle is 60°. Find the contents of the vessel between the vertex and the sphere by means of the formula, $V=\int\int\int dx\,dy\,dz$.

229. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Solve the differential equation $d^2y/dx^2=axy$.

MECHANICS.

192. Proposed by WILLIAM HOOVER, Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

A solid sphere rolls down a trough formed by two planes which make with each other an angle 2α. Find, by the principle of *vis viva*, the expression for the time of rolling down the trough when the inclination of the trough to the horizon is β.

193. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Three light smoothly jointed rods stand like a tripod—the three edges of a regular tetrahedron. A rectangular board, weight w, stands on this like an easel. Find the thrust on the rod which does not touch the easel.

AVERAGE AND PROBABILITY.

179. Proposed by HENRY HEATON, Atlantic, Iowa.

Through every point of the circumference of a given circle, chords are drawn in every possible direction. What is their average length?

180. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

There are n numbers in a box numbered from 1 to n. A number is drawn and replaced n times. Show that on the average the number of repeats is $\left(\frac{n-1}{n}\right)^n n$.

GROUP THEORY.

15. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Prove that there is no simple group of odd order less than 10,394.

A CORRECTION.

Professor G. Peano has called my attention to the fact that in my article on the transcendence of e and π, Vol. 12, page 223, line 9, of the MONTHLY, Newton's formulas which should be $a_n s_1 + a_{n-1} = 0$, $a_n s_2 + a_{n-1} s_1 + 2a_{n-2} = 0$, etc., are cited incorrectly. This makes it necessary to introduce in the function $\phi(x)$ at the bottom of page 221, the factor $a_n{}^{np-1}$ and to modify the expressions on pages 222 and 223 in a corresponding way. The corrections apply to the proof of the transcendence of π, that for the transcendence of e requiring no change. The necessary modifications are to be found in Mr. Lennes' and my "Introduction to Infinitesimal Analysis" where the proof is reproduced. O. VEBLEN.

NOTES AND NEWS.

Mr. L. Weeks of Yale University has been appointed instructor in mathematics in Purdue University.

R. D. Carmichael has been appointed Professor of Mathematics in the Presbyterian College for Men at Anniston, Ala.

Prof. T. E. McKinney of Marietta College has accepted the Professorship of Mathematics in the Wesleyan University.

Dr. George Bruce Halsted has been appointed Professor of Mathematics in the State Normal School at Greeley, Colorado.

Mr. Paul Dorweiler and Mr. C. F. Hagenow have been appointed Instructors in Mathematics in Armour Institute of Technology.

Prof. Nelson L. Roray of the Utica (N. Y.) Free Academy, has been appointed Instructor in Mathematics in the Jersey City High School.

Dr. H. L. Coar, instructor in mathematics in the University of Illinois, has accepted the Professorship of Mathematics in Marietta College.

Dr. Sisam of the Annapolis Naval Academy, and Dr. Dodd of Iowa, have been appointed Instructors in Mathematics in the University of Illinois.

At Syracuse University, Mr. F. F. Decker has been advanced from Assistant in Mathematics to Instructor in Mathematics, and Mr. H. F. Hart was reappointed Assistant in Mathematics.

Dr. Glenn's connection with the MONTHLY ceased with the June-July number. We take this opportunity to extend to him our sincere thanks for the very efficient services rendered in conducting its affairs during the past year.

The June-July number of the MONTHLY was mailed the first week in September, the delay being caused by having to send proofs to Philadelphia. This issue was mailed three weeks late. We hope to have future issues mailed promptly on the 28th of each month, excepting July and August.

BOOKS.

On Finite Algebras. By Leonard Eugene Dickson. Presented to the Imperial Society of Science at Göttingen, by Herrn David Hilbert. Reprinted from the Bulletin of the Society. Pamphlet, 36 pages.

The author, in the introduction, states that the object of the article is the study of the independence of a set of postulates for a finite field of p^n elements, p being a prime, and cites as an illustration of his results two of the simpler finite algebras whose elements form a commutative group under addition, whose elements not equal to zero form a non-commutative group under multiplication, and which obey the left-hand, but not the right-hand, distributive law. To obtain an algebra whose elements form a commutative group under addition, with a law of multiplication which is commutative and distributive, but not associative, while division is always possible, he employs a linear algebra with coördinates in an arbitrary field F not having the modulus 2 and with units $1, i, j$ with a multiplication table in which $i \times i = j$, $i \times j = b + \beta i$, $j \times i = b + \beta i$, $j \times j = -\beta^2 - 8bi - 2\beta j$, b and β being any marks of F such that $x^3 - \beta x - b$ is irreducible in F.

The greater part of the investigation deals with the determination of all finite algebras of the two types illustrated, for each type of which the number of elements must be a power of a prime. The determination of all such algebras for n odd is made exhaustive, subject, however, to the validity of the following theorem in abstract group theory:

Any group of order $p^n - 1$, p a prime and n an odd integer >1, contains a self-conjugate sub-group of order a power of a prime q, where q divides $p^n - 1$, but not $p^m - 1$, $m < n$.

This theorem the author verified through a very wide range of values. A demonstration of this theorem is desired. B. F. F.

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THE CONIC SECTIONS IN THE OLD JAPANESE MATHEMATICS.

By T. HAYASHI, Lecturer of Mathematics in the Tokyo Higher Normal School, Tokyo, Japan.

In a brief Note in the MONTHLY, Vol. XII, 1905, p. 166, it is remarked that "the most surprising fact about the old Japanese mathematics is that, while the most elementary parts were regarded as common property, the more advanced results were regarded as secrets which should be communicated to a very few." In fact, at most, one of the sons of one head master and two of his most highly versed disciples might be the keepers of the secrets of advanced mathematics as the successors of the head master. This curious fact, a parallel to which can be found in the Pythagorean school, interrupted the development of Japanese mathematics. But it became gradually impossible already before the Restoration of the Imperial government from the shogoon's authorities in 1868, to keep secret among a very few the proofs of theorems and the solutions of problems which they obtained from their predecessors or which they invented themselves.

Takakazu Seki (1642—1708) was the founder of the most famous school. This school was so flourishing that the mathematics of that school is the representative of Japanese mathematics and the development of mathematics in Japan was almost entirely accomplished by the scholars of that school. Seki, a contemporary of Newton and Leibnitz, invented the method of calculation called the Enri method, which well resembles the infinitesimal calculus. The now-living head master of the Seki school, Mr. Chōrin Kawakita, turned over to the library of the Tokyo Imperial University printed books and manuscript books belonging to him and containing theorems and problems kept in secret for about two hundred years. The number of these books together with those which the library bought reaches about two thousand, so that Prof. P. Harzer says that they are a